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Dispersion relations for zero-helicity gravitational waves in a relativistic expanding medium

J Plinio Baptista†§ and Daniel Gerbal‡

† Laboratoire de Gravitation et Cosmologie Relativistes and Universidade Federal do Espirito Santo, Brazil
‡ Laboratoire d'Astrophysique, Observatoire de Paris, 92190 Meudon, France

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Abstract. In this paper we present a study of evolution of helicity-0 waves (density waves) on the FRW cosmological background by using a two-time scale method for solving the perturbed field equation. The kinetic description is adopted by means of the collisionless Liouville equation self-consistently coupled with Einstein's equations. The wave packets obtained are solutions of the shift type. The effects of the expansion of the background (geometrical effects) are contained in these solutions and lead to a power law for the scale factor instead of an exponential law in time. The rate of growth is obtained from dispersion relations which are studied in the case of a cold gravitational plasma. For large wavenumber q the Newtonian dispersion relation is recovered as is the time behaviour of the Newtonian solution, and for vanishing q the behaviour of the relativistic hydrodynamics for the pressureless case is also recovered. It is found that in a relativistic treatment of the cold gravitational plasma, the smaller q becomes the faster the instability grows.

1. Introduction

This paper is devoted to the theory of zero-helicity waves in a general relativistic massive-particle medium. The asymptotic expansion method used to solve the equation describing the problem under study is the so-called two-time scale method.

In a previous paper (Asseo *et al* 1976) (to be referred to as paper I) it has been shown that the helicity representation ($\pm 2, \pm 1, 0$) of waves in a relativistic gravitational plasma arises naturally (indeed it is also true in an electromagnetic plasma). Dispersion relations for two-helicity waves—also called radiative waves—have been given in paper I. The zero-helicity waves we study are associated with so-called 'density waves'. It is clear that the knowledge of the growth of density waves in a Friedman–Robertson–Walker (FRW) Universe background is essential to the understanding of the presently observed inhomogeneous behaviour of the universe. This is connected with the still unsolved problem of the formation of galaxies, or clusters of galaxies. See for instance the review paper by Jones (1976).

This paper belongs, as paper I, to the large ensemble of works devoted to the theory of perturbations of cosmological models initiated by the paper of Lifshitz (1946) and the principal results are given in standard texts such as those of Weinberg (1972) and Peebles (1980).

§ Present address: Laboratoire de Physique Theorique, 11 rue Pierre et Marie Curie, 75231 Paris Cedex, France.

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However, in spite of the large number of papers in the field, this domain of study is far from closed. For instance Press and Vishniac (1980) and Bardeen (1980) have recently studied very large perturbations and some technical aspects such as 'gauge conditions'. The growth of peculiar inhomogeneities (black holes) has been considered by Carter (1979). This paper represents an effort to enlarge the understanding of these areas.

The majority of works of this kind are based on a hydrodynamic description of matter, although the use of kinetic theory has been so powerful in plasma theory. The first part of paper I was devoted to the study of the kinetic theory of a relativistic gravitational plasma (general formulation and two-helicity waves). Therefore, it is necessary to derive the zero-helicity dispersion relations, as well as to study waves with helicity one. A Vlasov system of equations is obtained from a BBGKY hierarchy neglecting all correlations. Haggerty and Severne (1976) have studied the stability of this equation and analysed the growth of instabilities in gravitational plasma.

Note that a comparison between a kinetic and a hydrodynamic description of waves is important.

We cannot disconnect the description of the problem at hand from the method of solution which we adopt. It should be emphasised that the two-time scale method is particularly well adapted to physical situations in which cosmic time plays some special role and it is an efficient tool to classify almost automatically the different approximations at each order. The existence of an expanding background gives rise to dispersion effects on the waves we are studying; however, the use of a two-time scale approximation allows for the separation of the expansion from dispersion effects. Therefore, we have solved equations for the four zero-helicity components up to the third order of approximations (i.e. terms such as $\ddot{S}$, $\dot{S}^2$ are neglected).

From the set of basic equations, the application of the two-time scale method (§ 3) leads to two hierarchies of equations which we analyse by means of a Fourier transform. In appendix 2 the gauge conditions we have chosen are explained. Each function has been expanded in a series of functions of order zero, one, two and so on. As shown in paper I, gravitational waves of order zero and of order one, apart from those of helicity two, vanish. Therefore, we relabelled our notation: the zeroth order in this paper corresponds to the order two in paper I.

Solutions are available (§ 4) under conditions which consist of a *matter* dispersion relation modified by a *geometrical* dispersion relation. We study how the solution obtained evolves in time. To do that we need to make an analysis of the dispersion relation.

2. The basic equations

Liouville's equation coupled self-consistently to Einstein equations in the Vlasov approximation is the tool that we choose to study the propagation of waves in a massive medium,

$$R_{\mu\nu}-\tfrac{1}{2}G_{\mu\nu}R=\chi\int \det(-\breve{g}_{\mu\nu})\mathcal{N}(x^\lambda,u^\lambda)u_\mu u_\nu\frac{\mathrm{d}_3u}{u_4} \tag{1}$$

$$u^\alpha\partial_\alpha\mathcal{N}(x^\lambda,u^\lambda)-\Gamma^\rho_{\alpha\beta}u^\alpha u^\beta\frac{\partial\mathcal{N}(x^\lambda,u^\lambda)}{\partial u^\rho}=0 \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor, $\check{g}_{\mu\nu}$ is the metric tensor, $\chi = 8\pi G/c^2$ the Einstein gravitational constant, $\Gamma^{\rho}_{\alpha\beta}$ the Christoffel symbol and $\mathcal{N}(x^{\lambda}, u^{\lambda})$ the one-particle distribution function. The background medium is supposed to be homogeneous and isotropic in a sufficiently large region.

2.1. *The metric tensor* $\check{g}_{\mu\nu}$

The metric tensor $\check{g}_{\mu\nu}$ is split into two parts:

$$\check{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \tag{3}$$

$g_{\mu\nu}$ is the background metric tensor. Instead of using the Robertson–Walker metric (making $c = 1$ and $k = 0$)

$$\mathrm{d}s^2 = \mathrm{d}t'^2 - S^2(t')\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{4}$$

as the background metric we shall make a change of time variable as

$$\mathrm{d}t' = S(t)\,\mathrm{d}t. \tag{5}$$

This will facilitate the subsequent calculations in view of the application of the two-time scale method as will be shown in § 3.1. Our background metric will be

$$g_{\mu\nu} = S^2(t)\eta_{\mu\nu} \tag{6}$$

where $\eta_{\mu\nu} = (1, -\delta_{ij})$.

The only non-vanishing connection components for the background are

$$\Gamma^i_{j4} = \Gamma^4_{ij} = (\dot{S}/S)\delta_{ij} \qquad \Gamma^4_{44} = \dot{S}/S \tag{7}$$

which gives the following components of the curvature tensor

$$R_{ijkm} = \dot{S}^2(\delta_{ik}\delta_{jm} - \delta_{jk}\delta_{im}) \qquad R_{4i4j} = -(S\ddot{S} - \dot{S}^2)\delta_{ij} \tag{8}$$

and the Ricci tensor

$$R_{ij} = \left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2}\right)\delta_{ij} \qquad R_{44} = 3\left(-\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2}\right) \qquad R = -6\frac{\ddot{S}}{S^3} \tag{9}$$

where δ_{ij} is the Krönecker tensor.

In the helicity representation of the perturbed metric tensor $h_{\mu\nu}$, we are interested only in the zero-helicity components (see table 1)

$$\begin{aligned} f &= h_{11} - \tfrac{1}{3}(h_{11} + h_{22} + h_{33}) && \text{'spin' 2} \\ g &= h_{14} && \text{'spin' 1} \end{aligned} \tag{10a}$$

Table 1. Variables $H(\ ,\ )$ characterised by a definite spin and helicity, given as a function of the components $h_{\mu\nu}$ of the weak gravitational field.

Spin \ Helicity	2	1	0
2	$H\pm(2,2) = \frac{1}{2}(h_{22} - h_{33}) \pm \mathrm{i}h_{23}$	$H\pm(2,1) = h_{12} \pm \mathrm{i}h_{13}$	$H(2,0) = h_{11} - 1/3h_{ll} = f$
1		$H\pm(1,1) = h_{42} \pm \mathrm{i}h_{43}$	$H(1,0) = h_{41} = g$
0			$H^T(0,0) = h_{44} = h$
			$H^S(0,0) = h_{ll} = H$

and

$$h = h_{44} \qquad H = h_{11} + h_{22} + h_{33} \qquad \text{'spin' } 0. \tag{10b}$$

Therefore, we are only concerned with the following components: h_{44}, h_{41}, h_{11} and $h_{22} = h_{33}$ which are equal by symmetry. Note that we have in fact chosen $x^1 = x$ as the direction of propagation of the waves of perturbations.

The perturbed connection is reduced to

$$\begin{aligned}
\chi^4_{44} &= \frac{1}{2S^2}\left(\dot{h}_{44} - 2\frac{\dot{S}}{S}h_{44}\right) & \chi^4_{4i} &= \frac{1}{2S^2}\left(\partial_i h_{44} - 2\frac{\dot{S}}{S}h_{4i}\right) \\
\chi^4_{ij} &= \frac{1}{2S^2}\left(\partial_i h_{j4} + \partial_j h_{i4} - \dot{h}_{ij} - 2\frac{\dot{S}}{S}h_{44}\delta_{ij}\right) & \chi^i_{44} &= \frac{1}{2S^2}\left(\partial_i h_{44} - \dot{h}_{4i} + 2\frac{\dot{S}}{S}h_{4i}\right) \\
\chi^i_{4j} &= \frac{1}{2S^2}\left(\partial_i h_{j4} - \partial_j h_{4i} - \dot{h}_{ij} + 2\frac{\dot{S}}{S}h_{ij}\right) & \chi^i_{jk} &= \frac{1}{2S^2}\left(\partial_i h_{jk} - \partial_j h_{ki} - \partial_k h_{ij} + 2\frac{\dot{S}}{S}h_{4i}\delta_{jk}\right)
\end{aligned} \tag{11}$$

(a superscript dot means a time derivative and $\partial_i \equiv \partial/\partial x^i$). The only components of the de Rham–Lichnerowicz operator are

$$\begin{aligned}
\Delta h_{44} &= \frac{1}{S^2}\left[\Box h_{44} + \left(4\frac{\ddot{S}}{S} - 10\frac{\dot{S}^2}{S^2}\right)h_{44} + 4\frac{\dot{S}}{S}\delta^{mn}\partial_m h_{n4} - 2\frac{\dot{S}}{S}\dot{h}_{44} + \left(2\frac{\ddot{S}}{S} - 4\frac{\dot{S}^2}{S^2}\right)\delta^{mn}h_{mn}\right] \\
\Delta h_{i4} &= \frac{1}{S^2}\left[\Box h_{i4} - 2\frac{\dot{S}}{S}\dot{h}_{i4} + 2\frac{\dot{S}}{S}\delta^{mn}\partial_m h_{ni} + \left(4\frac{\ddot{S}}{S} - 8\frac{\dot{S}^2}{S^2}\right)h_{i4} + 2\frac{\dot{S}}{S}\partial_i h_{44}\right] \\
\Delta h_{ij} &= \frac{1}{S^2}\left[\Box h_{ij} - 2\frac{\dot{S}}{S}\dot{h}_{ij} + 2\frac{\dot{S}}{S}(\partial_i h_{4j} + \partial_j h_{i4})\right. \\
&\qquad \left. + \left(2\frac{\ddot{S}}{S} - 4\frac{\dot{S}^2}{S^2}\right)h_{44}\delta_{ij} - 2\frac{\dot{S}^2}{S^2}\delta^{mn}h_{mn}\delta_{ij} + 4\frac{\dot{S}^2}{S^2}h_{ij}\right]
\end{aligned} \tag{12}$$

where

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \sum_1^3 \frac{\partial^2}{\partial x^{i2}}.$$

2.2. *The Vlasov system*

The Vlasov system of coupled equations for the background is

$$R_{\mu\nu} = \chi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) \tag{13}$$

$$u^\mu \partial_\mu N(x^\lambda, u^\lambda) - \Gamma^\alpha_{\rho\sigma} u^\rho u^\sigma \partial N(x^\lambda, u^\lambda)/\partial u^\alpha = 0 \tag{14}$$

$$T^{\mu\nu} = \int \sqrt{-g}N(x^\lambda, u^\lambda)u^\mu u^\nu \frac{\mathrm{d}_3 u}{u_4}. \tag{15}$$

$N(x^\lambda, u^\lambda)$ is the one-particle distribution function for the background and $T^{\mu\nu}$ is the energy–momentum tensor. We have the energy density

$$\rho = T^4{}_4 \tag{16}$$

and

$$-p = T^1{}_1 = T^2{}_2 = T^3{}_3 \tag{17}$$

is the pressure for an isotropic distribution function.

The linearisation of the Vlasov system, equations (1), (2), for the perturbation has been given by Droz-Vincent and Hakim (1968) (see also paper I). In our case it can be written

$$(\mathscr{L}h)_{\mu\nu} \equiv \Delta h_{\mu\nu} - h_{\mu\nu}S^{\lambda}_{\lambda} - g_{\mu\nu}h_{\alpha\beta}S^{\alpha\beta} + 2(h_{\mu\alpha}S^{\alpha}_{\nu} + h_{\nu\alpha}S^{\alpha}_{\mu}) + \Lambda_{\mu\nu} = \Sigma_{\mu\nu} \tag{18}$$

$$u^{\alpha}\partial_{\alpha}Z(x^{\lambda}, u^{\lambda}) - \Gamma^{\alpha}_{\rho\sigma}u^{\rho}u^{\sigma}\frac{\partial Z}{\partial u^{\alpha}}(x^{\lambda}, u^{\lambda}) - \chi^{\alpha}_{\rho\sigma}u^{\rho}u^{\sigma}\frac{\partial N}{\partial u^{\alpha}}(x^{\lambda}, u^{\lambda}) \tag{19}$$

where $\Lambda_{\mu\nu} = \nabla_{\mu}I_{\nu} + \nabla_{\nu}I_{\mu}$, $I_{\mu} = \nabla_{\rho}(h^{\rho}_{\mu} - \frac{1}{2}\delta^{\rho}_{\mu}\bar{h})$, ∇ is the covariant derivative with respect to the metric of the background and $\bar{h} = g^{\alpha\beta}h_{\alpha\beta}$. We have

$$(\mathscr{L}h)_{44} = \frac{1}{S^2}\Big[\Box h_{44} - 2\delta^{mn}\partial_m\dot{h}_{n4} + \ddot{h}_{44} + \delta^{mn}\ddot{h}_{mn} + \frac{\dot{S}^2}{S}(-\dot{h}_{44} + 10\delta^{mn}\partial_m h_{n4} - 3\delta^{mn}\dot{h}_{mn})$$
$$+\left(2\frac{\ddot{S}}{S} - 17\frac{\dot{S}^2}{S^2}\right)h_{44} + \left(4\frac{\ddot{S}}{S} - 5\frac{\dot{S}^2}{S^2}\right)\delta^{mn}h_{mn}\Big]$$

$$(\mathscr{L}h)_{ij} = \frac{1}{S^2}\Big[\Box h_{ij} - \delta^{mn}\partial_i\partial_m(h_{nj} + \tfrac{1}{2}\delta_{nj}h_{44} - \tfrac{1}{2}\delta_{nj}\delta^{rs}h_{rs})$$
$$-\delta^{mn}\partial_j\partial_m(h_{ni} + \tfrac{1}{2}\delta_{ni}h_{44} - \tfrac{1}{2}\delta_{ni}\delta^{rs}h_{rs}) + \partial_i\dot{h}_{4j} + \partial_j\dot{h}_{i4}$$
$$+2(\dot{S}/S)[2\partial_i h_{4j} + 2\partial_j h_{i4} - \dot{h}_{ij} + (\delta^{mn}\partial_m h_{n4} - \tfrac{1}{2}\dot{h}_{44} - \tfrac{1}{2}\delta^{rs}\dot{h}_{rs})\delta_{ij}]$$
$$+\left(2\frac{\ddot{S}}{S} - 5\frac{\dot{S}^2}{S^2}\right)h_{44}\delta_{ij} + 2\frac{\ddot{S}}{S}h_{ij} - \left(2\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2}\right)\delta^{rs}h_{rs}\delta_{ij}\Big] \tag{20}$$

$$(\mathscr{L}h)_{i4} = \frac{1}{S^2}\Big[\Box h_{i4} - \partial_i(\delta^{mn}\partial_m h_{n4} - \tfrac{1}{2}\dot{h}_{44} - \tfrac{1}{2}\delta^{rs}\dot{h}_{rs})$$
$$-\delta^{mn}\partial_m(\dot{h}_{in} + \tfrac{1}{2}\delta_{ni}\dot{h}_{44} - \tfrac{1}{2}\delta_{in}\delta^{rs}\dot{h}_{rs}) + \ddot{h}_{i4} + 2\frac{\dot{S}}{S}(3\partial_i h_{44}$$
$$-2\dot{h}_{i4} + 3\delta^{mn}\partial_m h_{in} - \delta^{mn}\partial_i h_{mn}) + \left(4\frac{\ddot{S}}{S} - 14\frac{\dot{S}^2}{S^2}\right)h_{i4}\Big].$$

The right-hand side of equation (18) is given by

$$\Sigma_{\mu\nu} = -2\chi\Big(\int\sqrt{-g}(u_{\mu}u_{\nu} - \tfrac{1}{2}g_{\mu\nu})Z(x^{\lambda}, u^{\lambda})\frac{d_3u}{u_4}$$
$$+\frac{1}{2S^2}(h_{44} - h_{11} - h_{22} - h_{33})(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)\Big) \tag{21}$$

where $Z(x^{\lambda}, u^{\lambda})$ is the perturbed distribution function.

The analysis of stability against perturbations of zero-helicity modes is nothing but the solutions of the system of equations (18), (19). Clearly, the field and matter of the background system appear in it. They are related by equations (13), (14) and (15). This system is solved elsewhere (Hakim 1968, Bel 1969) and it is well known that there are no stationary solutions (other than trivial or exotic ones). Thus, approximation methods are necessary to solve the system (18) and (19). These methods generally lead to some splitting between the background field and background matter. However, from the foundations of general relativity theory, matter and field are intimately related. Therefore in obtaining the solutions, it is insufficient to take into account only the matter or only the field. The correct approach would treat matter and field on the same footing.

3. Splitting of geometry against matter interaction effects

To solve equations (18) and (19) we want to make an approximation such that matter and geometry would be, step by step, disconnected. The time scale which characterises evolution of the background, and consequently the evolution of the scale factor $S(t)$, is the Hubble time scale

$$T_{\rm H} = (\dot{S}/S)^{-1} = (3/\chi\rho)^{1/2}. \tag{22}$$

However, we study wave phenomena, particularly instability phenomena, in which matter is dominant. Thus another scale of time related to these waves arises: T_i the time scale of the order of a few periods of oscillations, or, more precisely, the duration of validity of the linear theory. Because, as we shall see in §§ 5 and 6, this theory cannot account for large densities which occur in configurations such as stars, galaxies or clusters of galaxies, etc which implies the existence of nonlinear phenomena, then T_i is smaller than the age of the Universe. Therefore,

$$T_i = \varepsilon T_{\rm H} \qquad \varepsilon \ll 1; \tag{23}$$

this feature gives rise to the two-time scale approximation.

3.1. Asymptotic expansion

Applied to phenomena evolving in an FRW background, the asymptotic procedure, now well known (Cole 1968, Nayfeh 1973, see also paper I), needs several steps described as follows.

(1) A change of time as appears in equation (5). The reason (explained in § 3 paper I, or in Cole 1968, p 102) actually is a non-mixing between the two time scales.

(2) The use of the fundamental ansatz given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial \mathcal{T}} + \varepsilon \frac{\partial}{\partial T} \tag{24}$$

$$\chi(t) = \chi(\mathcal{T}, T) = \chi_{(0)}(\mathcal{T}, T) + \varepsilon\chi_{(1)}(\mathcal{T}, T) + \varepsilon^2\chi_{(2)}(\mathcal{T}, T) + \mathrm{O}(\varepsilon^2) \tag{25}$$

where $\mathcal{T}$ and T are the 'times' measured respectively in units of T_i and $T_{\rm H}$, and $\chi(t)$ is any function which appears in our equations.

(3) After resolution of the two hierarchical systems of equations which arise (short time scale and large time scale systems), we rewrite the functions as functions of cosmological times. Note that the method—up to first order—is equivalent to a WKB method, but it allows a quite automatic classification of quantities inferred at each order.

The equations (18), (19), after the asymptotic expansion give rise to two systems.

System 1

$$\begin{aligned} &\frac{1}{S^2}\bar{\Box} h_{(r)\mu\nu} + \mathrm{O}(\varepsilon^2) = \Sigma_{(r)\mu\nu} + \mathrm{O}(\varepsilon^2) \\ &u^\alpha \partial_\alpha Z_{(r)}(x^\lambda, u^\lambda) - \chi^\rho_{(r)\alpha\beta} u^\alpha u^\beta \frac{\partial N}{\partial u^\rho}(x^\lambda, u^\lambda) + \mathrm{O}(\varepsilon^2) = 0 \end{aligned} \tag{26}$$

where $\bar{\Box}$ is the d'Alembertian operator ($r=0, 1$)

$$\bar{\Box}\equiv\frac{\partial^2}{\partial\mathcal{T}^2}-\sum_1^3\frac{\partial^2}{\partial x^{i2}}$$

and

$$\Sigma_{(r)\mu\nu}=-2\chi\int Z_{(r)}(x^\lambda, u^\lambda)(u_\mu u_\nu-\tfrac{1}{2}g_{\mu\nu})\sqrt{-g}\,\frac{d_3u}{u_4} \tag{27}$$

and the $h_{\mu\nu}$ reduce to the components given in § 2 (χ above is the Einstein gravitational constant and must not be confused with $\chi(t)$ of (25) nor with $\chi^{\rho}_{\alpha\beta}$, the perturbation of the Christoffel symbols). Note that at zero and first order matter equations (system 1 above) are similar, therefore the subscript is unnecessary.

System 2

$$\begin{aligned}
&\frac{\partial^2}{\partial T\,\partial\mathcal{T}}H_{(0)}(\mathcal{T}, x, T)+\frac{\partial^2}{\partial T\,\partial x}g_{(0)}(\mathcal{T}, x, T)-\frac{\dot{S}}{S}\left(\frac{\partial H_{(0)}}{\partial\mathcal{T}}(\mathcal{T}, x, T)-4\frac{\partial g_{(0)}}{\partial x}(\mathcal{T}, x, T)\right)=0\\
&\frac{\partial^2}{\partial T\,\partial T}H_{(0)}(\mathcal{T}, x, T)+6\frac{\partial^2}{\partial T\,\partial x}g_{(0)}(\mathcal{T}, x, T)+\frac{\partial^2}{\partial T\,\partial x}h_{(0)}(\mathcal{T}, x, T)\\
&\qquad+\frac{\dot{S}}{S}\left(\frac{4}{3}\frac{\partial H_{(0)}}{\partial x}(\mathcal{T}, x, T)+4\frac{\partial f_{(0)}}{\partial x}(\mathcal{T}, x, T)+8\frac{\partial h_{(0)}}{\partial x}(\mathcal{X}, x, T)\right)=0\\
&3\frac{\partial^2}{\partial T\,\partial\mathcal{T}}f_{(0)}(\mathcal{T}, x, T)+2\frac{\partial^2}{\partial T\,\partial x}g_{(0)}(\mathcal{T}, x, T)\\
&\qquad-\frac{\dot{S}}{S}\left(3\frac{\partial f_{(0)}}{\partial\mathcal{T}}(\mathcal{T}, x, T)-8\frac{\partial g_{(0)}}{\partial x}(\mathcal{T}, x, T)\right)=0\\
&\frac{\partial^2}{\partial T\,\partial\mathcal{T}}H_{(0)}(\mathcal{T}, x, T)+3\frac{\partial^2}{\partial T\,\partial\mathcal{T}}h_{(0)}(\mathcal{T}, x, T)\\
&\qquad+\frac{\dot{S}}{S}\left(2\frac{\partial h_{(0)}}{\partial\mathcal{T}}(\mathcal{T}, x, T)+4\frac{\partial g_{(0)}}{\partial x}(\mathcal{T}, x, T)\right)=0
\end{aligned} \tag{28}$$

with the gauge conditions

$$\begin{aligned}
&\frac{\partial}{\partial x}f_{(r)}(\mathcal{T}, x, T)+\frac{1}{2}\frac{\partial}{\partial x}h_{(r)}(\mathcal{T}, x, T)-\frac{1}{6}\frac{\partial}{\partial x}H_{(r)}(\mathcal{T}, x, T)-\frac{\partial}{\partial\mathcal{T}}g_{(r)}(\mathcal{T}, x, T)=0\\
&\frac{\partial}{\partial x}g_{(r)}(\mathcal{T}, x, T)-\frac{1}{2}\frac{\partial}{\partial\mathcal{T}}h_{(r)}(\mathcal{T}, x, T)-\frac{1}{2}\frac{\partial}{\partial\mathcal{T}}H_{(r)}(\mathcal{T}, x, T)=0.
\end{aligned} \tag{29}$$

3.2. *The Fourier analysis*

When using the two-time scale method we have hypothesised that functions under consideration are functions of two independent times. However, some functions such as the cosmological scale factor $S(\mathcal{T}, T)=S(T)$ are only large-scale time dependent. Therefore they are constant functions in the short time scale. The Robertson–Walker metric is then a Minkowskian one. This is the reason why system 1 is easily written in a compact form; for example the d'Alembertian used in equation (26), but where the fourth coordinate is related to $\mathcal{T}$, i.e. $\partial_4\equiv\partial/\partial\mathcal{T}$. In view of solving systems 1 and

2, we define the Fourier transform of a function $\chi(x^\alpha, T)$ as

$$\chi(x^\alpha, T) = \int \chi(k^\rho, T)\exp(\mathrm{i}k_\mu x^\mu)\,\mathrm{d}^4k \tag{30}$$

with $x^\mu = (\mathcal{T}, x^i)$, $k_\mu = (\omega, q)$ and $k_\mu x^\mu = \omega\mathcal{T} + qx$. Note that $\chi(x^\alpha, T)$ is also a function of T.

3.2.1. System 1, after the use of (24), becomes

$$g^{\alpha\beta}k_\alpha k_\beta h_{\rho\sigma}(k^\lambda, T) = 2\chi \int Z(k^\lambda, T)(u_\rho u_\sigma - \tfrac{1}{2}g_{\rho\sigma})\sqrt{-g}\,\frac{\mathrm{d}_3u}{u_4} \tag{31}$$

$$Z(k^\alpha, T) = \frac{1}{2k_\alpha u^\alpha}\,p^\lambda_{\rho\sigma}(k^\alpha, T)\,\frac{\partial N}{\partial u^\lambda}(k^i, u^\lambda, T) \tag{32}$$

$$p^\alpha_{\rho\sigma}(k^\lambda, T) = k_\rho h^\alpha_\sigma(k^\lambda, T) + k_\sigma h^\alpha_\rho(k^\lambda, T) - k^\alpha h_{\rho\sigma}(k^\lambda, T). \tag{33}$$

The treatment of this type of equation is well known (see paper I), and leads to the form

$$\mathcal{D}^{\rho\sigma}_{\alpha\beta}\,h_{\rho\sigma}(k^\lambda, T) = 0 \tag{34}$$

with

$$\begin{aligned}\mathcal{D}^{\rho\sigma}_{\alpha\beta} \equiv k^2\delta^{(\rho}_\alpha\delta^{\sigma)}_\beta + \chi[&2T^{(\rho}_\alpha\delta^{\sigma)}_\beta + 2T^{(\rho}_\beta\delta^{\sigma)}_\alpha - (k_\alpha I^{\rho\sigma}_\beta + k_\beta I^{\rho\sigma}_\alpha) + 2T_{\alpha\beta}g^{\rho\sigma}\\ &-(k^\rho I^\sigma_{\alpha\beta} + k^\sigma I^\rho_{\alpha\beta}) + k^2 J^{\rho\sigma}_{\alpha\beta} - g_{\alpha\beta}(Tg^{\rho\sigma} - k^{(\rho}I^{\sigma)} + \tfrac{1}{2}k^2J^{\rho\sigma})]\end{aligned} \tag{35}$$

where $(\rho\sigma)$ means symmetrisation in this index and $k^2 \equiv g^{\alpha\beta}k_\alpha k_\beta$. Two types of integral functions appear naturally

$$I^{\alpha\beta\ldots}_\rho \equiv \int N(x^i, u^\lambda, T)\,\frac{u^\alpha u^\beta u_\rho}{k_\lambda u^\lambda}\sqrt{-g}\,\frac{\mathrm{d}_3u}{u_4} \tag{36}$$

$$J^{\alpha\beta\ldots}_{\rho\sigma\ldots} \equiv \int N(x^i, u^\lambda, T)\,\frac{u^\alpha u^\beta u_\rho u_\sigma}{(k_\lambda u^\lambda)^2}\sqrt{-g}\,\frac{\mathrm{d}_3u}{u_4} \tag{37}$$

where $I^{\alpha\beta\ldots}_\rho$ has an odd number of indices while $J^{\alpha\beta\ldots}_{\rho\sigma\ldots}$ has an even number. Symmetries of these functions are explained in appendix 1.

3.2.2. The same Fourier transform (30) is applied to the system 2 (equation (25)), but in this case the dependence $\mathcal{T}$ and x (i.e. ω or q) is assumed to be known (from the system 1). Then the set of equations can be written as

$$\begin{aligned}&\frac{\partial}{\partial T}H_{(0)}(k^\lambda, T) + \frac{q}{\omega}\frac{\partial}{\partial T}g_{(0)}(k^\lambda, T) = \frac{\dot S}{S}\Big(H_{(0)}(k^\lambda, T) - \frac{4q}{\omega}g_{(0)}(k^\lambda, T)\Big)\\ &\frac{\partial}{\partial T}H_{(0)}(k^\lambda, T) + \frac{6\omega}{q}\frac{\partial}{\partial T}g_{(0)}(k^\lambda, T) + \frac{\partial}{\partial T}h_{(0)}(k^\lambda, T)\\ &\qquad = -\frac{\dot S}{S}[\tfrac{4}{3}H_{(0)}(k^\lambda, T) + 4f_{(0)}(k^\lambda, T) + 8h_{(0)}(k^\lambda, T)]\\ &\frac{2q}{\omega}\frac{\partial}{\partial T}g_{(0)}(k^\lambda, T) + 3\frac{\partial}{\partial T}f_{(0)}(k^\lambda, T) = \frac{\dot S}{S}\Big(3f_{(0)}(k^\lambda, T) - \frac{8q}{\omega}g_{(0)}(k^\lambda, T)\Big)\\ &\frac{\partial}{\partial T}H_{(0)}(k^\lambda, T) + 3\frac{\partial}{\partial T}h_{(0)}(k^\lambda, T) = -\frac{\dot S}{S}\Big(\frac{4q}{\omega}g_{(0)}(k^\lambda, T) + 2h_{(0)}(k^\lambda, T)\Big).\end{aligned} \tag{38}$$

4. Solutions

The following procedure is adopted in view of obtaining the general solution.

(i) We solve first the short-scale time equations (26) (matter equations). The matter dispersion relations are derived from this solution.

(ii) We solve the large-scale time equations (38) (geometrical equations) in the next step.

As usual, at first order, amplitude modifications appear which represent the large-scale time evolution. We strongly emphasise that the geometrical equations have been solved but only gauge-compatible solutions are retained.

4.1. Matter dispersion relation—cold case

The matrix given by (35) is generally very complicated even in the cold case. The integrals $I_\rho^{\alpha\beta}$ and $J_{\rho\sigma}^{\alpha\beta}$ are written in this case

$$I_\rho^{\alpha\beta}=\frac{n_0 m\bar{u}^\alpha\bar{u}^\beta\bar{u}_\rho}{k_\lambda\bar{u}^\lambda}\qquad J_{\alpha\beta}^{\rho\sigma}=\frac{n_0 m\bar{u}^\rho\bar{u}^\sigma\bar{u}_\alpha\bar{u}_\beta}{(k_\lambda\bar{u}^\lambda)^2}. \tag{39}$$

The non-vanishing integrals are listed in appendix 1. n_0 is the density number and m is the mass of the particles of the gravitational gas.

Then, in helicity notation, system 1 is represented by the matrix system

$$(\omega^2-q^2)f_{(0)}(k^\lambda,T)\equiv\mathscr{D}_1 f_{(0)}(k^\lambda,T)=0 \tag{40}$$

$$\begin{vmatrix} -(\omega^2-q^2)/6\omega_\mathrm{p}^2 & (\omega^2-q^2)/2\omega_\mathrm{p}^2+2 & 0 \\ (\omega^2-q^2)/6\omega_\mathrm{p}^2-1 & (\omega^2-q^2)/2\omega^2 & q/\omega \\ 0 & -q/\omega & (\omega^2-q^2)/2\omega_\mathrm{p}^2+2 \end{vmatrix}$$
$$\times\begin{vmatrix} H_{(0)}(k^\lambda,T) \\ h_{(0)}(k^\lambda,T) \\ g_{(0)}(k^\lambda,T)\end{vmatrix}=\mathscr{D}_2\begin{vmatrix} H_{(0)}(k^\lambda,T) \\ h_{(0)}(k^\lambda,T) \\ g_{(0)}(k^\lambda,T)\end{vmatrix}=0. \tag{41}$$

The plasma frequency $\omega_\mathrm{p}=\frac{1}{2}\chi n_0 mS^2=4\pi G n_0 mS^2$ is chosen to be unity. The solutions are of the form

$$\chi_i(\mathscr{T},x,T)=\int\chi_i(\omega,q,T)\exp[\mathrm{i}(\omega\mathscr{T}+qx)]\delta(\mathscr{D}_i)\,\mathrm{d}\omega\,\mathrm{d}q \tag{42}$$

where δ is Dirac's distribution, and

$$\det\mathscr{D}_1=\omega^2-q^2=0 \tag{43}$$

$$\det\mathscr{D}_2=\omega^8-3(q^2-1)\omega^6+(3q^4-7q^2-28)\omega^4-(q^6-5q^4-28q^2+96)\omega^2-q^6=0 \tag{44}$$

are the two dispersion relations; $\chi_i(\mathscr{T},x,T)$ being any of the four components. However, note that f waves decouple from the others.

4.2. Geometrical equations

Solutions of equation (38) are not very difficult to find. However, the compatibility of these solutions with the gauge conditions is not so easy, and tedious calculations are necessary. We want to point out that solutions can be obtained but only when the matter dispersion relations (43) and (44) are taken into account.

The solutions depend on *two* (and only two) constants. Thus we have,

$$\begin{aligned}
H_{(0)}(\omega, q, T) &= C_2(\omega, q)S^{-3/2}(T) + \tfrac{2}{3}C_1(\omega, q) \\
g_{(0)}(\omega, q, T) &= (\omega/q)(C_2(\omega, q)S^{-3/2}(T) + \tfrac{1}{6}C_1(\omega, q)) \\
h_{(0)}(\omega, q, T) &= C_2(\omega, q)S^{-3/2}(T) - \tfrac{1}{3}C_1(\omega, q) \\
f_{(0)}(\omega, q, T) &= \tfrac{2}{3}C_2(\omega, q)S^{-3/2}(T) + \tfrac{4}{9}C_1(\omega, q).
\end{aligned} \tag{45}$$

4.3. General solutions

We obtain the general solutions following the usual procedure of the two-time scale method, adding each order of solutions. After reconstituting the initial time, by using equation (24), the solutions are, as they should be, wave packets. We obtain two 'modes'.

Mode I

$$\begin{aligned}
H(t, x) &= 4\int qC_{\mathrm{I}}(q)\exp(\mathrm{i}qx)\exp\left(\mathrm{i}\int \omega(q, t)\,\mathrm{d}t\right)\mathrm{d}q \\
g(t, x) &= \int \omega(q)C_{\mathrm{I}}(q)\exp(\mathrm{i}qx)\exp\left(\mathrm{i}\int \omega(q, t)\,\mathrm{d}t\right)\mathrm{d}q \\
h(t, x) &= -2\int qC_{\mathrm{I}}(q)\exp(\mathrm{i}qx)\exp\left(\mathrm{i}\int \omega(q, t)\,\mathrm{d}t\right)\mathrm{d}q \\
f(t, x) &= \frac{8}{3}\int qC_{\mathrm{I}}(q)\exp[\mathrm{i}(qx + \omega' t)]\,\mathrm{d}q
\end{aligned} \tag{46}$$

Mode II

$$\begin{aligned}
H(t, x) &= 6S^{-3/2}(t)\int qC_{\mathrm{II}}(q)\exp(\mathrm{i}qx)\exp\left(\mathrm{i}\int \omega(q, t)\,\mathrm{d}t\right)\mathrm{d}q \\
g(t, x) &= 6S^{-3/2}(t)\int \omega C_{\mathrm{II}}(q)\exp(\mathrm{i}qx)\exp\left(\mathrm{i}\int \omega(q, t)\,\mathrm{d}t\right)\mathrm{d}q \\
h(t, x) &= 6S^{-3/2}(t)\int qC_{\mathrm{II}}(q)\exp(\mathrm{i}qx)\exp\left(\mathrm{i}\int \omega(q, t)\,\mathrm{d}t\right)\mathrm{d}q \\
f(t, x) &= 4S^{-3/2}(t)\int qC_{\mathrm{II}}(q)\exp[\mathrm{i}(qx + \omega' t)]\,\mathrm{d}q
\end{aligned} \tag{47}$$

ω', ω are the zeros of the dispersion relation (43), (44).

5. Analysis of the time dependence of the solutions

An easy way to understand the time dependence of the solutions we have obtained is to look at the behaviour of the density contrast

$$\Delta \equiv \delta\rho/\rho = K_4^4/T_4^4. \tag{48}$$

From (19), we obtain

$$\Delta_{\rm I} \propto S^{-1}(t) \exp\left(\int_{t_0}^{t} \gamma(q, t)\, dt\right) \tag{49}$$

$$\Delta_{\rm II} \propto S^{-5/2}(t) \exp\left(\int_{t_0}^{t} \gamma(q, t)\, dt\right) \tag{50}$$

I, II characterising the two modes. $\gamma(q, t)$ is the imaginary part of $\omega(q, t)$ arising from the dispersion relations (43), (44). Note that in (19), it is the cosmic time coming from the FRW cosmology that we used.

Clearly, the time dependence occurs from the variation of the amplitude of the wave packets (which comes from the geometrical influence of the background) and from the instantaneous phase $\gamma(q, t)$ (which characterises matter effects). It is now useful to re-introduce the dimensional phase

$$\gamma(q, t) = \gamma(q_1)\omega_{\rm p}(t). \tag{51}$$

The gravitational plasma frequency $\omega_{\rm p}$ is related to the Hubble 'constant' via the Einstein equations for the background $\omega_{\rm p}(t) = \sqrt{\frac{3}{2}}\dot{S}/S$. Then the density contrast becomes

$$\Delta_{\rm I} \propto S^{-1}(t) S^{\gamma(q)\sqrt{\frac{3}{2}}}(t) \tag{52}$$

$$\Delta_{\rm II} \propto S^{-5/2}(t) S^{\gamma(q)\sqrt{\frac{3}{2}}}(t); \tag{53}$$

this emphasises strongly but simply how the mathematical characterisation of an expanding background (S function of time) as compared with a static background (S constant) gives a power law for the scale factor.

The rate $\gamma(q)\sqrt{3/2} - 1$ of amplification for the less 'evanescent' mode depends on the wavenumber q, but only through $\gamma(q)$. Therefore, some comments on the dispersion relations are in order.

There are two dispersion relations; the first one, equation (43), is related to the f waves which are disconnected from other components. The second dispersion relation, equation (44), is devoted to the other components.

It should be noted that the f waves are not affected by any dispersive or damping effects. The case of relation (44) is obviously less simple. It is an equation of eighth degree leading to eight roots. We may question the different behaviour of our general relativistic case (giving eight modes) compared with the Newtonian case which gives rise to only one mode (Jean's mode). Jean's dispersion relation is obtained as a Newtonian limit of our general relativistic matter dispersion relation (43). The Newtonian approximation of the general theory of relativity may be made by making two simultaneous and necessary assumptions: (i) small-velocities condition, (ii) physical system not too large; i.e., respectively $\omega \ll 1$ and $q \gg 1$. Then (43) reduces to $\omega^2 + 1 = 0$. The eight roots of (43) are distributed as follows.

Mode a, defined by $\omega = \pm\omega_1(q)$; there are no damping or amplification effects.

Mode b, defined by $\omega = \pm\omega_2(q) \pm {\rm i}\gamma_2(q)$, ingoing and outgoing waves with damping or amplification.

Mode c, defined by $\omega = \pm {\rm i}\gamma_3(q)$: only damping or amplification effects.

The behaviour of these modes is described in figure 1 in which the ω curves are given, and in figure 2 in which the rates γ_2 and γ_3 of instabilities are given.

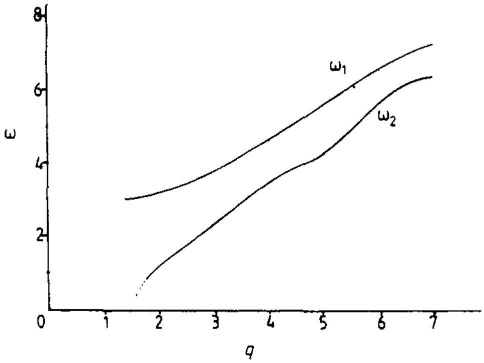

Figure 1. Real part of roots of dispersion relations ω is in units of $(4\pi Gmn_0)^{1/2}$. q is in units of ω/c.

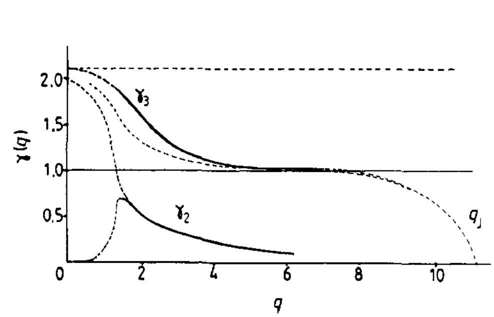

Figure 2. Imaginary part of the roots of dispersion relations. Same units as in figure 1. The full straight line located at $\gamma(q) = 1$ is the 'Newtonian cold' rate of growth. The broken straight line located at $\gamma(q) \sim 2.10$ is the hydrodynamics rate of growth.

For $q \lesssim 1.5$, γ_2 has two branches which meet giving a unique γ_2. Note that ω_2 vanishes for $q \lesssim 1.5$. γ_2 is generally very small (even smaller than γ_3) and tends asymptotically to 0.05.

Waves described by mode c are *not* progressive waves. Then they give rise to an *absolute instability*. Therefore following Bers and Briggs (1976) other modes should be without any real physical significance. Thus the value of $\partial\omega_2/\partial q$, which is often greater than 1 ($=c$ in our units), is also without physical significance. The term γ_3 is larger and tends asymptotically to 1, for large wavenumbers. We emphasise that $\gamma_3 \to 1$ is the Newtonian Jean 'dispersion' curves for a *cold* gravitational plasma. It is the graphic translation of the Newtonian limit of our relativistic relation of dispersion as explained before.

In contrast to the Newtonian Jean dispersion relation, which in the cold case is independent of q, the relativistic one depends essentially on q. We can easily suppose how our γ_3 curve will be modified when a small temperature would be taken into account: knowing that the Newtonian behaviour is recovered for large q. On the other hand, the Newtonian γ curve vanishes for the wavenumbers q where $q \gtrsim q_J$ (q_J is the Jean wavenumber) and then the pressure effect will counterbalance the gravitational effect. Then it is reasonable to suppose that the pressure effects when they are sufficiently important would act in such a way that the γ_3 curves would grow more slowly than they would in the cold case, for $q \to 0$.

6. Conclusion

This paper is devoted to the study of the zero-helicity waves (density waves) on the FRW background. To solve the Einstein–Liouville systems of equations describing the collisionless gravitational plasma we have used the two-time scale expansion method.

This method appears to be, in the relativistic case (paper I and herein) as with the Newtonian case (Baptista and Gerbal 1980), structurally well adapted for separating phenomena governed by different time scales. Technically speaking, it allows for a very simple treatment of the dispersion relation because quantities involved depending only on the large scale time are treated as constant in the first step. After returning to the cosmological time, these quantities are automatically time functions in the phase of a WKB-type exponential.

We obtain a dispersion relation which takes into account matter effects and as well geometrical effects due to the interaction of the waves with the matter and with the non-flat and non-stationary background. The matter part of the dispersion relation is dependent upon the wavenumber q (figures 1 and 2). This is not true for the geometrical part (at least to the order for which we solve our equations). Therefore, the study of the instability for the collisionless case shows that the rate actually depends upon q, *even in the cold case.* This is the difference between our kinetic relativistic theory and both the Newtonian and the hydrodynamics treatments. For the latter two treatments in the cold case the Jean instability does not depend on q. Note that here $\gamma_3(q)$ tends to about 2.12 for small q, i.e. the hydrodynamics result is recovered (as it must be) in the hydrodynamic limit! However, although this is satisfactory for the coherence of our work, this limiting value is obtained out of the domain of validity of the two-time scale method, i.e. for lengths of the orders of the particle horizon.

The way in which the exponential instability, characterising a static plasma, is transformed into a *power law of the scale factor* $S(t)$ in the non-stationary FRW background appears *clearly* (note that the power law for time is a consequence of the power time dependence for the scale factor).

The rate of growth of instability that we have computed recovers the Newtonian rate for large q. We have found solutions of our equations up to order three (although relabelled as order 1 in this paper) because geometrical effects appear there. Matter effects have been obtained to orders two and three. The next orders would give phase-shifted modifications, but unfortunately the Lorentz-like gauge condition is no longer practical after third order. It would be necessary to find another gauge condition. Nevertheless, it is the Lorentz-like gauge which ensures the classification of the field $h_{\alpha\beta}$ following helicity. Therefore calculations at higher orders for the asymptotic expansion seem to us to be very difficult and with few improvements added to the present value calculated because it appears clearly that the main time feature of the solutions is obtained up to order three. In our minds an effort must be made in another direction: the difference between an exponential growth and a power law is probably not so dramatic as is often stated. The duration of the validity of the linear 'regime' is not very large. During this time the exponential growth on an instability is not large compared with the power law growth of an instability. This emphasises the absolute necessity of studying the full nonlinear instability.

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Appendix 1.

If we take into account that the four-vector k_λ does not depend on the velocity u^α, we can readily see that

$$k_\rho I^{\rho\beta}_\alpha = T^\beta_\alpha \qquad k_\rho I^\rho = T \tag{A1.1}$$

where T is the trace of $T_\alpha{}^\beta$; $T_\alpha{}^\beta$ being the energy–impulsion tensor of the background given by equation (15). We also have the following relation

$$k_\rho J^{\rho\sigma} = I^\sigma \qquad k_\rho k_\sigma J^{\rho\sigma\beta}_\alpha = T_\alpha{}^\beta. \tag{A1.2}$$

Now following our choice of the propagation direction of our perturbation and taking into account that $N(x^\lambda, u^\lambda)$ is an even function of the three-velocity, the symmetry properties of the integrals $I^{\alpha\beta}_\rho$, $J^{\alpha\beta}_\rho$ and $J^{\alpha\beta}_{\rho\sigma}$ are the same as the products $u^\alpha u^\beta u_\rho$ and $u^\alpha u^\beta u_\rho u_\sigma$ for α and β equal to 2 or 3. In this case, the integrals vanish whenever they are odd in the index 2 or 3 (each integration is taken on the interval $-\infty$ to $+\infty$).

For the cold case we put $N(x^\lambda, u^\lambda) = \rho(x^\lambda)\delta(u^\alpha - \bar{u}^\alpha)$ and in this case (36) and (37) give (39). The non-vanishing integrals are

$$I^4 = I^{44}_4 = \rho(x^\lambda)/\omega \qquad \text{and} \qquad J^{44} = J^{44}_{44} = \rho(x^\lambda)/\omega^2 \tag{A1.3}$$

with $\rho(x^\lambda) = mn_0(x^\lambda)$.

Appendix 2.

The complete Einstein tensor (background plus variation) is divergence free. The splitting of the metric tensor given by equation (3) leads to the following identity

$$\nabla_\mu \delta S^{\mu\nu} + \chi^\nu_{\mu\sigma} S^{\mu\sigma} + \chi^\mu_{\mu\sigma} S^{\sigma\nu} \equiv 0 \tag{A2.1}$$

i.e. the perturbed Einstein tensor is *not* divergence free with respect to the background metric. The energy–momentum tensor, constructed from a distribution function which is a solution of the Liouville equation, is also divergence free with respect to the total metric. The same splitting (equation (3)) leads to

$$\nabla_\mu K^{\mu\nu} + \chi^\nu_{\mu\sigma} T^{\mu\sigma} + \chi^\mu_{\mu\sigma} T^{\sigma\nu} \equiv 0. \tag{A2.2}$$

We remark now that the variation of the variation of the connection, $\chi^\rho_{\alpha\beta}$, is $O(\varepsilon^2)$ in the asymptotic expansion. Therefore,

$$\nabla_\mu \delta S^{\mu\nu} + O(\varepsilon^2) \equiv \nabla_\mu K^{\mu\nu} + O(\varepsilon^2) \equiv 0. \tag{A2.3}$$

Therefore, the de Donder gauge condition

$$I_\mu = \nabla_\rho (h^\rho_\mu - \tfrac{1}{2}\delta^\rho_\mu \bar{h}) = 0$$

used in the paper by Droz-Vincent and Hakim (1968) and used in our paper is only compatible with the perturbed Einstein equations for either the zero- and first-order approximation or for an empty background.

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